International Journal of Theoretical Physics, Vol. 12, No. 6 (1975), pp. 393–400

Minkowski Space with Order Topology is Simply Connected

S. NANDA

Mathematics Department, Regional Engineering College, Rourkela 769008 (Orissa), India

and

H. K. PANDA

Mathematics Department, Rairangpur College, Orissa, India

Received: 16 *August* 1974

Abstract

Among the large number of topologies which have been suggested for Minkowski space, the order topology, i.e., the one generated by the positive cone at the origin and its translates, turns out to be most peculiar; yet it has some very pleasant properties. For example, it is pathwise connected and not arcwise connected and every loop based at a point is homotopic to the constant loop at that point; in other words, Minkowski space with the order topology is simply connected.

1. *Notation and Terminology*

Since Zeeman (1967) suggested the 'fine topology' for Minkowski space, the space-time continuum of Special Relativity, several authors, Nanda (1969, 1971, 1972), Whiston (1972) and Vroegindeweij (1973), have suggested new topologies satisfying the requirement that the homeomorphism group of each topology is either the group G, generated by the inhomogeneous Lorentz group together with dilatations, or its subgroup G_0 consisting of elements which are order-automorphisms. All these topologies with the exception of the order topology (definitions follow later) are finer than the Euclidean topology and are therefore Hausdorff. They are also pathwise connected and not simply connected. The object of this paper is to consider the simplest of all possible topologies on Minkowski space, i.e., the order topology (Vroegindeweij, 1973), and point out its peculiarities. In spite of its peculiarities, Minkowski space with this topology has a very pleasant property that it is simply connected.

In what follows, M will denote Minkowski space, i.e., $M = \{(x_0, x_1, x_2, x_3): x_i$ are reals$\}$ together with the quadratic form Q:

$$Q(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

We have a positive cone K at the origin defined by

$$K = \{x \in M : Q(x) > 0, x_0 > 0\}$$

which gives rise to the usual partial order on M as follows:

$$x < y \Leftrightarrow y - x \in K$$

Closely associated with the cone K, we have another positive cone L at the origin defined by

$$L = \{x \in M : Q(x) \geqslant 0\, x_0 > 0\}$$

which again defines another partial order on M:

$$x \leqslant y \Leftrightarrow y - x \in L$$

For each $x \in M$, we can have the translates of the positive cones K and L.

$$K + x = \{y + x : y \in K\}$$

and

$$L + x = \{y + x : y \in L\}$$

which we shall denote by $K(x)$ and $L(x)$ respectively. We also define for each $x \in M$ the cones $K^*(x)$ and $L^*(x)$ as follows:

$$K^*(x) = K(x) \cup \{x\}$$

and

$$L^*(x) = L(x) \cup \{x\}$$

It is easy to check that

$$K^*(x) = \{y : x < y \text{ or } x = y\}$$

and, similarly,

$$L^*(x) = \{y : x \leqslant y \text{ or } x = y\}$$

The cones $\{K^*(x) : x \in M\}$ generate a topology on M which we call the order topology (Whiston calls it the Zeeman-order topology). For convenience, we shall henceforth call it the K-topology on M and denote the corresponding topological space by M^K.

As usual, Int (A), $\bar{A}$ and A^c will mean the interior, closure and complement, respectively, of the set A with respect to the topology under consideration.

2. *Properties of the K-Topology*

The K-topology has very poor separability properties. It is easy to see that it is not Hausdorff, it is not even T_1. For example, if $x < y$ then any open set about x will contain y. It is clearly T_0. It is also not compact though it is locally compact because every basic open set $K^*(x)$ is a compact neighbourhood of the point x. It is both connected and locally connected.

A peculiar topological property of the K-topology which we shall use later is the following.

Proposition 2.1. If

$$x \in M, \qquad \overline{\{x\}} = \{y : y < x, \text{ or } y = x\}$$

Proof. Since $\bar{A} = [\text{Int}(A^c)]^c$, it will be worthwhile considering the set $\text{Int}(\{x\}^c)$, i.e., the interior of the set $M - \{x\}$. Note that if $y \nless x$ and $y \neq x$, then $K^*(y) \subset M - \{x\}$, thus making y an interior point of $M - \{x\}$. Conversely, these are the only interior points of $M - \{x\}$, for, if $y < x$ or $y = x$, then $K^*(y)$ will contain x and therefore y cannot be an interior point of $M - \{x\}$. Thus $\text{Int}(M - \{x\}) = \{y \in M : y \nless x, y \neq x\}$ or, equivalently, $[\text{Int}(M - \{x\})]^c = \{y \in M : y < x \text{ or } y = x\}$ and the proof is complete.

Since any basic open set of the K-topology is of the form $K^*(x)$, it is easy to prove that any continuous map f from M^K into itself is order-preserving, i.e., $x < y \Rightarrow f(x) < f(y)$. Similarly, if the map f is one-one then f^{-1} is continuous if and only if f^{-1} is order-preserving. Thus the homeomorphisms of M^K are precisely the one-one maps of M^K onto itself which are order-automorphisms. A direct application of Zeeman's theorem (Zeeman, 1964) will then yield the following (we, however, omit the proof since it involves no new techniques).

Proposition 2.2. The homeomorphism group of M^K is G_0.

Moreover, M^K has also the following interesting property:

Proposition 2.3. M^K is superconnected i.e., every open set is connected.

Proof. Let O be an arbitrary open set in M^K and assume to the contrary that it is disconnected. Then $O = O_1 \cup O_2$ where $O_1 \cap O_2 = \emptyset$ and O_1 and O_2 are non-void open sets in O. Since O itself is open in X, it follows that O_1 and O_2 are also open in X. Choose $x \in O_1$ and $y \in O_2$ arbitrarily. (This is possible since O_1 and O_2 are non-void.) From the very definition of the topology we have $x \in K^*(x) \subset O_1$ and $y \in K^*(y) \subset O_2$. Note that it is possible to choose $z \in M$ such that $x < z$ and $y < z$; thus $K^*(x) \cap K^*(y) \neq \emptyset$. Therefore $O_1 \cap O_2 \neq \emptyset$, and this gives a contradiction. The proof is therefore complete.

Corollary 1. It is also clear from this proof that any two non-void open sets of M^K will have non-void intersection.

We shall next prove the following

Proposition 2.4. M^K is pathwise connected.

Proof. Let x and y be arbitrary points in M; choose a point $z \in K^*(x) \cap K^*(y)$. We shall first show that x and z are pathwise connected. Define $f: [0, 1] = I \mapsto M^K$ as follows:

$$f(t) = \begin{cases} z & \text{for } 0 \leqslant t < 1 \\ x & \text{for } t = 1 \end{cases}$$

Note that $f^{-1}(K^*(z)) = [0, 1)$ and $f^{-1}(K^*(x)) = [0, 1]$; hence for any open set O_z and O_x containing z and x respectively, $f^{-1}(O_z) = [0, 1)$ if $x \notin O_z$; $f^{-1}(O_z) = [0, 1]$ if $x \in O_z$ and $f^{-1}(O_x) = [0, 1]$. Thus $f: I \to M^K$ is continuous with $f(0) = z$ and $f(1) = x$. Similarly, y and z are pathwise connected and therefore x and y are pathwise connected.

One of the nicest properties of Zeeman's fine topology is the existence of order-preserving paths. (A path f is said to be order-preserving if and only if $t_1 < t_2 \Rightarrow ft_1 < ft_2$ where t_1 and $t_2 \in I$ and $t_1 < t_2$ means the usual order on the real line.) Such paths have been interpreted as the paths of freely moving particles. In contrast to this, we have the following.

Proposition 2.5. No path $f: I \to M^K$ is order-preserving.

Proof. Suppose on the contrary that $f: I \to M^K$ is continuous and that $t_1 < t_2 \Rightarrow ft_1 < ft_2$ for every pair of points t_1 and t_2 in I. Let $t \in (0, 1)$. Continuity of f implies that $f^{-1}(K^*(ft))$ is an open set about t. Therefore, there is an $\epsilon > 0$ such that $(t - \epsilon, t + \epsilon) \subset f^{-1}(K^*(ft))$ or, equivalently, $f(t - \epsilon, t + \epsilon) \subset K^*(ft)$. If we choose a point $t' \epsilon (t - \epsilon, t)$ then we have $t' < t$ and $ft' \in K^*(ft)$ i.e., $ft < ft'$, thus giving a contradiction to our assumption that f is order-preserving.

We shall now make a simple application of the Baire category theorem (which states that a complete metric space cannot be expressed as a countable union of nowhere dense sets) to show that M^K is not arcwise connected. (A topological space X is said to be arcwise connected iff for every pair of points x and y there is a continuous one-one map from the unit interval to X such that $f(0) = x$ and $f(1) = y$.) In fact we shall show that no continuous map from I to M^K can be one-one.

Proposition 2.6. If $f: I \to M^K$ is a continuous map, then there is a point $t_0 \in I$ and an interval V with $t_0 \in V \subset I$ such that $f(V) = ft_0$.

Proof. Let $t \in I$. Continuity of f implies that $f^{-1}(K^*(ft))$ is an open set in I containing t. Therefore it is possible to choose a real number $\theta_t > 0$ such that $(t - \theta_t, t + \theta_t) \subset f^{-1}(K^*(ft))$ or, equivalently, $f(t - \theta_t, t + \theta_t) \subset K^*(ft)$. Thus, corresponding to every $t \in I$, we have a real number $\theta_t > 0$. This enables us to define a real-valued function h on I as follows: $h(t) = \theta_t$. Note that h satisfies the following two conditions: (i) $h(t) > 0$ for every $t \in I$ and (ii) $f(t - ht, t + ht) \subset K^*(ft)$.

Define $A_n = \{t \in I : h(t) > 1/n\}$. Since $h(t) = \theta_t > 0$ for every $t \in I$, it follows that every $t \in I$ is an element of some A_n. Thus $I = \cup_{n=1}^{\infty} A_n$. I, being a complete metric space, belongs to the second category and therefore it cannot be expressed as a countable union of nowhere dense sets. Consequently at least one of these A_n's, say A_m, is not nowhere dense, i.e., Int $(\bar{A}_m) \neq \emptyset$, where Int (A) denotes the interior of the set A and bar denotes closure with respect to the usual topology on I. Int $(\bar{A}_m)$ is an open set and hence a countable union of disjoint open intervals; choose an interval $U \subset$ Int $(\bar{A}_m)$. Let $U \cap A_m = B_m$; obviously then B_m is dense in U. Note moreover that for every $t \in B_m$, $h(t) > 1/m$.

Choose a point $t_0 \in B_m$. Since U is open and t_0 is an interior point of U, there is a $k > 0$ such that $(t_0 - k, t_0 + k) \subset U$. Since a smaller $k > 0$ will also satisfy this inclusion relation, we choose k such that $k < \frac{1}{2}m$. Let $V = (t_0 - k, t_0 + k)$ and $C_m = V \cap B_m$, then C_m is dense in the interval V.

We have now $C_m \subset B_m \subset A_m$. Therefore, for every $t \in C_m, h(t) > 1/m$; consequently, $f[t - 1/m, t + 1/m] \subset K^*(ft)$. Moreover, since $k < \frac{1}{2}m$, it follows that $f[t - k, t + k] \subset K^*(ft)$. But $t \in C_m$ is arbitrary, hence $f(V) \subset K^*(ft)$ for every $t \in C_m$.

We now claim that f maps the entire interval V to a single point. First, suppose to the contrary that there exist t_1 and t_2 in C_m such that $t_1 \neq t_2$ with $f(t_1) \neq f(t_2)$. We will then have the following three cases to consider.

(i) If $ft_1 < ft_2$, then $f(V) \subset K^*(ft_1) \cap K^*(ft_2) = K^*(ft_2)$ which is impossible for $ft_1 \notin K^*(ft_2)$.

(ii) Similarly it is impossible to have $ft_2 < ft_1$.

(iii) Finally, if ft_1 and ft_2 are not comparable, i.e. neither $ft_1 < ft_2$ nor $ft_2 < ft_1$, then we shall have $f(V) \subset K^*(ft_1) \cap K^*(ft_2)$ which is again not possible for $K^*(ft_1) \cap K^*(ft_2)$ will contain neither ft_1 nor ft_2.

Thus it is impossible to have t_1 and t_2 in C_m with $t_1 \neq t_2$ and $ft_1 \neq ft_2$; therefore all points of C_m are mapped to a single point ft_0. It will now be enough to show that other points of V are also mapped to the same point.

Note that since f is continuous, $f(\overline{C_m}) \subset \overline{f(C_m)}$ i.e., $f(V) \subset \overline{\{ft_0\}}$. This inclusion, together with the inclusion $f(V) \subset K^*(ft_0)$, imply that $f(V) \subset K^*(ft_0) \cap \overline{\{ft_0\}}$. By Proposition 2.1, we have $\overline{\{ft_0\}} = \{y \in M : y < ft_0$ or $y = ft_0\}$; on the other hand, $K^*(ft_0) = \{y \in M : ft_0 < y$ or $ft_0 = y\}$. Thus, $f(V) \subset K^*(ft_0) \cap \overline{\{ft_0\}} = \{ft_0\}$ and the proof is complete.

3. *The Fundamental Group of M^K*

In spite of the pathological nature of the K-topology, M^K has the most interesting property that it is simply connected. We shall now prove the following

Proposition 3.1. Any loop f at a point x is homotopic (Greenburg, 1967) to the constant loop e_x at the point x.

Proof. Let f be a loop at the point $x \in M$, i.e., $f: I \to M^K$ is a continuous map with $f(0) = f(1) = x$. Continuity of f implies that $f(I)$ is compact. Therefore the open cover $\{K^*(f(p))\}_{p \in I}$ for $f(I)$ admits a finite subcover $\{K^*(f(p_i))\}_{i=1,2,\ldots,n}$. Choose $a \in M$ with $a < f(p_i)$ for every $i \in \{1, 2, \ldots, n\}$. This means that $K^*(f(p_i)) \subset K^*(a)$ for every i. We therefore have $f(I) \subset \cup_{i=1}^{n} K^*(f(p_i)) \subset K^*(a)$.

We will now construct a continuous function $F: I \times I \to M^K$ such that $F(0, t) = f(t)$ and $F(1, t) = e_x(t) = x$ for every $t \in I$ and $F(s, 0) = F(s, 1) = x$ for every $s \in I$. This will show that the loop f is homotopic to the constant loop e_x defined by $e_x(t) = x$ for every $t \in I$. Note that since $f: I \to M^K$ is continuous, $f^{-1}(K^*(x))$ is an open set in I; call it O_x. $I - O_x$ is then a closed set in I. We now define the continuous map $F: I \times I \to M^K$ as follows:

$$F(s,t)=\begin{cases} f(t) & \text{for } s\in[0,1/2), t\in I\\ x & \text{for } s=1/2, t\in O_x\\ a & \text{for } s=1/2, t\in I-O_x\\ e_x(t) & \text{for } s\in(1/2,1], t\in I\end{cases}$$

From the construction, it is clear that $F(0,t)=f(t), F(1,t)=e_x(t); F(s,0)=x$ and $F(s,1)=x$, $\forall\, s\in I$. It remains, however, to show that F, as defined above, is continuous. It is obvious that $F(I\times I)=f(I)\cup\{a\}$. Since $F(I)\subset K^*(a)$ by our choice of a, it follows that $F^{-1}(K^*(a))=I\times I$. Furthermore, for any point $z\in f(I), f^{-1}(K^*(z))=O_z$ is an open set in I. We shall now consider two cases, namely when (i) $z<x$ and when (ii) $z\nless x$. In the first case $K^*(z)\supset K^*(x)$; in the second case, $x\notin K^*(z)$. In the first case, $F^{-1}(K^*(z))=\{[0,1/2)\times O_z\}\cup\{I\times O_x\}\cup\{(1/2,1]\times I\}$ which is a union of three open sets in $I\times I$ and is therefore open. In the second case, when $x\notin K^*(z)$, we have $F^{-1}(K^*(z))=[0,1/2)\times O_z$ which is again an open set in $I\times I$. Thus in any case, the inverse image by F of basic open sets in M^K are open in $I\times I$ and this proves our assertion that F is continuous.

Combining Proposition 2.4 and Proposition 3.1, we have the following

Theorem. M^K is simply connected.

4. *Final Remarks*

It is not very hard to see that the K-topology is not comparable to the Euclidean topology. Its poor separability properties indicate that it has 'too few' open sets. One would then expect such a topology to be minimal in the sense that no strictly weaker topology on M can have the homeomorphism group G_0. As the following example will indicate, such an assumption is false.

Example. It is easy to see that the family of sets $\{L^*(x)\}_{x\in M}$ generates a topology on M. Let us call it the L-topology and denote the corresponding topological space by M^L. We claim that the K-topology is strictly finer than the L-topology. To prove this assertion, note first of all that for every basic open set $L^*(x)$ of the L-topology, $L^*(x)=\cup\{K^*(z):z\in L^*(x)\}$; consequently, every L-open set is K-open. Conversely, it is obvious that no $K^*(x)$ is an open set in the L-topology. Thus the L-topology is strictly weaker than the K-topology. That the homeomorphism group of the L-topology is also G_0 is easy to prove. We therefore have the following.

Proposition 4.1. The K-topology on M is not minimal.

Most of the topological properties of the K-topology are also shared by the L-topology. For example, one can have the same constructions to prove that M^L is superconnected, pathwise connected (and not arcwise connected) and simply connected. Much of this situation relating the order structure with the topological structure can also be put in a more general setting and one can have the same sort of results as proved in this paper.

Acknowledgments

The authors wish to thank Professor S. P. Misra for many valuable discussions and one of us (H.K.P.) wishes to thank the Government of India for financial help through its Research Training Scholarship Scheme.

Note Added in Proof

Dr. Panda's present address is Mathematics Department, Ravenshaw College, Cuttack-3 (Orissa), India.

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